

A LOCAL METHOD FOR POSETS

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ABSTRACT. We propose some conditions on a poset that produce a small chain complex for its homology. This allows to compare simplicial complexes and Quillen's complexes under the same prism. It turns out they differ in the existence or not of free faces in an acyclic complex.

1. INTRODUCTION

In this work we investigate a collection of posets that, roughly speaking, are posets whose initial rays are contractible in a homogeneous way. Among them we find (the incidence poset of) any abstract simplicial complex Δ and Quillen's complex $\mathcal{A}_\pi(G)$ of the non-trivial elementary abelian π -subgroups of a finite group G at the prime π [3]. In fact, in the setup we propose, simplicial complexes appear as the limit case of Quillen's complexes for the "prime" $\pi = 1$.

In order to introduce these notions, let \mathcal{P} be a poset, $p \in \mathcal{P}$ an object and consider the subposet $\mathcal{P}_{\leq p} = \{r \in \mathcal{P} | r \leq p\}$. This initial ray $\mathcal{P}_{\leq p}$ is contractible and hence acyclic as it contains the initial object p . Denote by $\hat{\mathcal{P}}$ the poset obtained by augmenting \mathcal{P} by a minimal element $\hat{0}$ and assume \mathcal{P} is graded by some integer valued increasing function \dim with $\dim \hat{0} = -1$. We shall say that the collection $\mathcal{K} = \{\mathcal{K}_p, \eta_p\}_{p \in \mathcal{P}}$, where \mathcal{K}_p is a subposet of $\hat{\mathcal{P}}_{\leq p}$ containing $\hat{0}$ and $\eta_p: \mathcal{K}_p \rightarrow \hat{\mathcal{P}}_{\leq p} \setminus \mathcal{K}_p$ is a map, is a *local covering family* for \mathcal{P} if

for all $p \in \mathcal{P}$ and all $r \in \mathcal{K}_p$ we have $r < \eta_p(r)$ and $\dim \eta_p(r) = \dim r + 1$

and the maps $\{\eta_p\}_{p \in \mathcal{P}}$ are compatible in certain way (see Definition 12 for full details). These conditions are similar to those of Stanley on decomposition of acyclic simplicial complexes [4, Theorem 1.2] and of Forman on Discrete Morse Theory for cell complexes [2]. They differ in that there the maps η_p are assumed to be *bijective*.

For instance, let Δ be a simplicial complex over the set of vertices $\{0, \dots, m\}$. For each $\sigma \in \Delta$ define \mathcal{K}_σ and η_σ by

$$\mathcal{K}_\sigma = \{\tau \subseteq \sigma | \sigma^* \notin \tau\} \text{ and } \eta_\sigma(\tau) = \tau \cup \{\sigma^*\},$$

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where $\sigma^* = \min \sigma$. Then $\mathcal{K}_\Delta = \{\mathcal{K}_\sigma, \eta_\sigma\}_{\sigma \in \Delta}$ is a local covering family for Δ . In this case the maps η_σ are bijections. The situation for Quillen's complex $\mathcal{A}_\pi(G)$ of the finite group G at the prime π is similar. Start by choosing a total order for the order- π subgroups of G : $\{V_0, \dots, V_m\}$. Then we can define a local covering family $\mathcal{K}_{\mathcal{A}_\pi(G)}$ for $\mathcal{A}_\pi(G)$ by setting for $H \in \mathcal{A}_\pi(G)$ the following:

$$\mathcal{K}_H = \{I \leq H \mid H^* \not\leq I\} \text{ and } \eta_H(I) = \langle I, H^* \rangle,$$

where $H^* = \min\{V_i \mid V_i \leq H\}$. Note that in this case the maps η_H are surjections.

It is not hard to see that similar local covering families can be constructed on any poset \mathcal{P} for which for all $p \in \mathcal{P}$ we have that $\mathcal{P}_{\leq p}$ is isomorphic to either a $(\dim p)$ -simplex or to $\mathcal{A}_\pi(C_\pi^{\dim p+1})$ for some prime π . We call such posets *locally simplicial* posets or *locally π -Quillen* posets respectively.

To further discuss the use of local covering families let $\Delta(\mathcal{P})$ denote the subdivision simplicial complex of a poset \mathcal{P} . Thus $\Delta(\mathcal{P})$ has n -simplices the compositions $p_0 < p_1 < \dots < p_n$ of objects of \mathcal{P} . If Δ is a simplicial complex let $|\Delta|$ denote its topological realization, whereas if \mathcal{P} is a poset define its realization as $|\mathcal{P}| = |\Delta(\mathcal{P})|$. For R a commutative ring with identity, write $C_*(\Delta; R)$ for the simplicial chain complex of the simplicial complex Δ (we omit the differential for clarity). So $C_n(\Delta; R)$ has one R -generator for each n -simplex of Δ . In particular, if \mathcal{P} is a poset we compute the homology of its realization via $H_*(|\mathcal{P}|; R) \cong H_*(C_*(\Delta(\mathcal{P}); R))$.

For the (incidence poset) $P = \Delta$ of the simplicial complex Δ , we have

$$H_*(|\mathcal{P}|; R) \cong H_*(C_*(\Delta(\mathcal{P}); R)) \cong H_*(C_*(\Delta; R))$$

and so we can compute $H_*(|\mathcal{P}|; R)$ using the smaller chain complex $C_*(\Delta; R)$ instead of the larger $C_*(\Delta(\mathcal{P}); R)$. This is not true in general as there is no such thing as $C_*(\mathcal{P}; R)$ for arbitrary \mathcal{P} . What we prove here is that if the poset \mathcal{P} is equipped with a local covering family \mathcal{K} then there is a chain complex $C_*^\mathcal{K}(\mathcal{P}; R)$ that does play the right role, i.e., it removes a subdivision.

1. Theorem. *Let \mathcal{P} be a graded poset with local covering family \mathcal{K} . Then there is a chain complex $C_*^\mathcal{K}(\mathcal{P}; R)$*

$$\dots \rightarrow \bigoplus_{p \in \mathcal{P}_i} R^{K_i^p} \rightarrow \dots \rightarrow \bigoplus_{p \in \mathcal{P}_1} R^{K_1^p} \rightarrow \bigoplus_{p \in \mathcal{P}_0} R \rightarrow 0$$

with explicit differential whose homology is $H_(|\mathcal{P}|; R)$.*

Here $\mathcal{P}_n = \{r \in \mathcal{P} \mid \dim r = n\}$, the objects of degree n . The numbers K_*^p depend upon \mathcal{K} and are defined inductively by $K_0^p = 1$ and $K_{n+1}^p = \sum_{q \in (\mathcal{K}_p)_n} K_n^q$. The description of the differential is given in Section 4. Next we investigate the geometric meaning of the local covering families \mathcal{K}_Δ and $\mathcal{K}_{\mathcal{A}_\pi(G)}$ defined above. In the former case, for any n -simplex σ of the simplicial complex Δ we have $K_n^\sigma = 1$. The geometric meaning of \mathcal{K}_Δ is explained by

2. Theorem. *Let Δ be a simplicial complex with local covering family \mathcal{K}_Δ . Then*

$$C_*^{\mathcal{K}_\Delta}(\Delta; R) \cong C_*(\Delta; R).$$

This means \mathcal{K}_Δ give us back the usual simplicial chain complex of Δ . For the covering family constructed for $\mathcal{A}_\pi(G)$, where G is a finite group and π is a prime, we have $K_n^H = \pi^{\frac{n(n+1)}{2}}$ for the subgroup $H \cong C_\pi^{n+1}$. Theorem 1 applied to $\mathcal{K}_{\mathcal{A}_\pi(G)}$ gives the following result.

3. Theorem. *Let G be a finite group and let π be a prime. Then there is a chain complex*

$$0 \rightarrow \bigoplus_{H \in \mathcal{A}_\pi(G)_{r-1}} R^{\pi \frac{r(r-1)}{2}} \rightarrow \dots \rightarrow \bigoplus_{H \in \mathcal{A}_\pi(G)_i} R^{\pi \frac{i(i+1)}{2}} \rightarrow \dots \rightarrow \bigoplus_{H \in \mathcal{A}_\pi(G)_0} R \rightarrow 0$$

with $r = rk_\pi(G)$ and with explicit differential whose homology is $H_*(|\mathcal{A}_\pi(G)|; R)$.

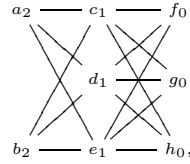
Here $rk_\pi(G)$ is the π -rank of G , i.e., the largest dimension of an elementary abelian π -subgroup of G . The following formula for the Euler characteristic of $|\mathcal{A}_\pi(G)|$ can be easily obtained via the Möbius function μ of the poset $\mathcal{A}_\pi(G)$ and the fact that $\mu(C_\pi^{n+1}) = \pi^{\frac{n(n+1)}{2}}$. In view of Theorem 3, this formula also appears now as the alternate sum of the dimensions of an appropriate chain complex.

4. Corollary. *Let G be a finite group and let π be a prime. Then:*

$$\chi(|\mathcal{A}_\pi(G)|) = \sum_{n=0}^{rk_\pi(G)-1} (-1)^n \pi^{\frac{n(n+1)}{2}} |\mathcal{A}_\pi(G)_n|.$$

A more exotic example of local covering family for a poset which is neither a simplicial complex nor a Quillen's poset follows. Its differential shall be described in Example 23.

5. Example. Consider the graded poset \mathcal{P} with 8 objects and shape



where the subindexes denote degree. The realization $|\mathcal{P}|$ has the homotopy type of a wedge of four spheres, $\bigvee_{i=1}^4 S_i^2$, and the following pairs define a local covering family \mathcal{K} for \mathcal{P} :

$$\begin{aligned} \mathcal{K}_a &= \{d, e, g, h, \hat{0}\}, \eta_a(d) = \eta_a(e) = a, \eta_a(g) = \eta_a(h) = c, \eta_a(\hat{0}) = f, \\ \mathcal{K}_b &= \{c, d, f, g, \hat{0}\}, \eta_b(c) = \eta_b(d) = b, \eta_b(f) = \eta_b(g) = e, \eta_b(\hat{0}) = h, \\ \mathcal{K}_c &= \{g, h, \hat{0}\}, \eta_c(g) = \eta_c(h) = c, \eta_c(\hat{0}) = f, \\ \mathcal{K}_d &= \{f, h, \hat{0}\}, \eta_d(f) = \eta_d(h) = c, \eta_d(\hat{0}) = g, \\ \mathcal{K}_e &= \{f, g, \hat{0}\}, \eta_e(f) = \eta_e(g) = c, \eta_e(\hat{0}) = h, \\ \mathcal{K}_f &= \{\hat{0}\}, \eta_f(\hat{0}) = f, \mathcal{K}_g = \{\hat{0}\}, \eta_g(\hat{0}) = g, \mathcal{K}_h = \{\hat{0}\}, \eta_h(\hat{0}) = h. \end{aligned}$$

The chain complex from Theorem 1 for \mathcal{K} and integral coefficients is

$$0 \longrightarrow \mathbb{Z}^4 \oplus \mathbb{Z}^4 \longrightarrow \mathbb{Z}^2 \oplus \mathbb{Z}^2 \oplus \mathbb{Z}^2 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0.$$

Compare the dimensions 8, 6 and 3 to the dimensions of $C_*(\Delta(\mathcal{P}); \mathbb{Z})$:

$$0 \longrightarrow \mathbb{Z}^{18} \longrightarrow \mathbb{Z}^{21} \longrightarrow \mathbb{Z}^8 \longrightarrow 0.$$

In order to compare the two general local covering families introduced so far, \mathcal{K}_Δ and $\mathcal{K}_{\mathcal{A}_\pi(G)}$, note first that the formulas for K_n^H and K_n^σ coincide if we let $\pi = 1$. Nevertheless, there are qualitative differences: call an object of the poset \mathcal{P} a *free object* if it is a non-maximal object which is contained in a unique maximal

object. So for simplicial complexes this is the usual concept of free face. It is well known that there exist simplicial complexes which are contractible but nevertheless they do not have a free face, e.g., any triangulation of the dunce hat. The situation for Quillen's posets, or more generally for locally π -Quillen posets, is quite the opposite:

6. Theorem. *Let \mathcal{P} be a locally π -Quillen poset at the prime π with $n = \max \dim \mathcal{P}$. If the homology group $H_n(|\mathcal{P}|; R)$ is zero then the proportion r of free objects among the non-maximal objects of \mathcal{P}_{n-1} satisfies:*

$$r \geq \frac{\pi^{n+1} - 2\pi^n + 1}{\pi^{n+1} - \pi^n}.$$

If we allow $\pi = 1$ in the bound of the statement it becomes 0, i.e., acyclic simplicial complexes may have no free faces. For actual primes π , $\pi \geq 2$, the bound is strictly larger than 0 and we deduce that there exists at least a free object in this case. This does not imply that we can inductively collapse $|\mathcal{P}| = |\Delta(\mathcal{P})|$ to $|\mathcal{P}_{\leq n-1}|$ as the subdivision operation does not preserve free objects of \mathcal{P} . Nevertheless, as $\pi \rightarrow \infty$, the ratio tends to 1^- , and hence, “asymptotically” on the prime π , a locally π -Quillen poset $|\mathcal{P}|$ of dimension n satisfying $H_n(|\mathcal{P}|; R) = 0$ can be collapsed into $|\mathcal{P}_{\leq n-1}|$.

Organization of the paper: In Section 2 we introduce preliminary notions, including a folklore chain complex for a graded poset, and certain notions of “suspension” and “truncation”. This is followed in Section 3 by the definition of local covering family and proof of Theorems 1 and 3. The study of the differential is postponed until Section 4, where Theorem 2 is also proven. The treatment of free objects is carried out in Section 5.

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2. SPHERICAL POSETS

A grading on a poset \mathcal{P} is a function $\dim: \mathcal{P} \rightarrow \mathbb{N}$ such that $p < q \implies \dim p < \dim q$ and such that 0 is attained. We also assume that $\dim p = \dim q + 1$ if p covers q . If $\dim \mathcal{P} \subseteq \mathbb{N}$ is finite we say that \mathcal{P} is bounded of dimension $\max \dim \mathcal{P}$. Every poset we consider is equipped with a grading unless stated otherwise. If \mathcal{P} is a poset we denote by $\hat{\mathcal{P}}$ the poset obtained by augmenting \mathcal{P} by a minimal element $\hat{0}$ for which $\dim \hat{0} = -1$ (and thus $\hat{\mathcal{P}}$ is formally speaking not a poset with a grading). Using an expression $\hat{\mathcal{P}}$ automatically means that $\hat{\mathcal{P}}$ is the augmentation of a graded poset \mathcal{P} . For $p, q \in \mathcal{P}$ members of a poset we use the notation $P_{<p} := \{r \in \mathcal{P} | r < p\}$, $P_{\leq p} := \{r \in \mathcal{P} | r \leq p\}$ and for an integer n we write $\mathcal{P}_{<n} := \{r \in \mathcal{P} | \dim(r) < n\}$, $\mathcal{P}_{\leq n} := \{r \in \mathcal{P} | \dim(r) \leq n\}$ and $\mathcal{P}_n := \{r \in \mathcal{P} | \dim(r) = n\}$.

Following Quillen [3], we say that a poset \mathcal{P} of dimension d is *d-spherical* if its nerve $|\mathcal{P}| = |\Delta(\mathcal{P})|$ is $(d-1)$ -connected, or equivalently, if it has the homotopy type of a bouquet of d -spheres. We say that the poset \mathcal{P} is *locally spherical* if $P_{<p}$ is $(\dim p - 1)$ -spherical for each $p \in \mathcal{P}$. Here we define a bouquet of -1 spheres as the empty set \emptyset .

Now let \mathcal{P} be a finite graded poset, let R be a commutative ring with identity, and let $C_*(\Delta(\mathcal{P}); R)$ the reduced simplicial chain R -complex whose homology is

$\tilde{H}_*(|\mathcal{P}|; R)$. The n -chains are given by

$$C_n(\Delta(\mathcal{P}); R) = \bigoplus_{p_0 < \dots < p_n} R$$

for $n \geq 0$ and by $C_{-1}(\Delta(\mathcal{P}); R) = R$. It is equipped with the usual differential $d = \sum (-1)^i d_i$ and the usual augmentation map ϵ . Here, for $0 \leq i \leq n$, the R -linear map $d_i: C_n(\Delta(\mathcal{P}); R) \rightarrow C_{n-1}(\Delta(\mathcal{P}); R)$ sends

$$p_0 < \dots < p_n \mapsto p_0 < \dots < p_{i-1} < p_{i+1} < \dots < p_n,$$

and the augmentation map sends $p_0 \in \mathcal{P}$ to $\epsilon(p_0) = 1 \in R$. The filtration of spaces

$$\emptyset \subseteq |\mathcal{P}_{\leq 0}| \subseteq \dots \subseteq |\mathcal{P}_{\leq i}| \subseteq |\mathcal{P}_{\leq i+1}| \subseteq \dots \subseteq |\mathcal{P}|$$

gives rise to the following increasing filtration of this chain complex

$$F_i C_j(\Delta(\mathcal{P}); R) = \bigoplus_{p_0 < \dots < p_j, \dim p_j \leq i} R$$

for $j \geq 0$ and by $F_i C_{-1}(\Delta(\mathcal{P}); R) = R$ for $i \geq -1$ and 0 otherwise. This filtration is bounded below and above and hence we have the following (folklore) convergent homological-type spectral sequence

$$E_{i,j}^1 = \tilde{H}_{i+j}(F_i C_{i+j}/F_{i-1} C_{i+j}) \Rightarrow \tilde{H}_{i+j}(|\mathcal{P}|; R).$$

Notice that $F_i C_{i+j}/F_{i-1} C_{i+j} = \bigoplus_{p_0 < \dots < p_{i+j}, \dim p_{i+j} = i} R$ which is 0 unless $j \leq 0$. Moreover, we have

$$\tilde{H}_{i+j}(F_i C_{i+j}/F_{i-1} C_{i+j}) \cong \tilde{H}_{i+j}\left(\bigvee_{\dim p=i} \Sigma |\mathcal{P}_{< p}|; R\right) \cong \bigoplus_{\dim p=i} \tilde{H}_{i+j-1}(|\mathcal{P}_{< p}|; R)$$

if $j \leq 0$ and 0 otherwise, where we define for convenience $\tilde{H}_{-1}(\emptyset; R) = R$. So, if \mathcal{P} is locally spherical, the spectral sequence degenerates to the chain complex

$$(7) \quad \dots \rightarrow \bigoplus_{\dim p=i} \tilde{H}_{i-1}(|\mathcal{P}_{< p}|; R) \rightarrow \dots \rightarrow \bigoplus_{\dim p=1} \tilde{H}_0(|\mathcal{P}_{< p}|; R) \rightarrow \bigoplus_{\dim p=0} R \rightarrow R.$$

We introduce, for a fixed object $p \in \mathcal{P}$ of dimension n , the linear maps “suspension at p ”, s_p , and “truncation at p ”, t_p :

$$\begin{array}{ccc} & s_p & \\ & \curvearrowright & \\ C_{n-1}(\Delta(P_{< p})) & & F_n C_n(\Delta(\mathcal{P})) \\ & \curvearrowleft & \\ & t_p & \end{array}$$

They are defined on basic elements by

$$(8) \quad \begin{aligned} s_p(p_0 < \dots < p_{n-1}) &= p_0 < \dots < p_{n-1} < p \text{ and} \\ t_p(p_0 < \dots < p_{n-1} < p_n) &= \begin{cases} p_0 < \dots < p_{n-1} & \text{if } p_n = p, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where $\dim p_i = i$. They posses the following properties:

$$(9) \quad \begin{aligned} d(s_p(z)) &= s_p(d(z)) + (-1)^n z, \\ d_i(t_p(z)) &= t_p(d_i(z)) \text{ for } i = 0, \dots, n-1, \text{ and} \\ t_{p'} s_p &= \begin{cases} \text{id} & \text{if } p' = p, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where p' is some object of dimension n . These maps induce the inverse to each other isomorphisms:

$$(10) \quad \bigoplus_{\dim p=n} \tilde{H}_{n-1}(|\mathcal{P}_{<p}|; R) \begin{array}{c} \xrightarrow{\sum_{\dim p=n} s_p} \\ \xleftarrow{\oplus_{\dim p=n} t_p} \end{array} \tilde{H}_n(F_n C_n / F_{n-1} C_n).$$

The chain complex (7) has homology equal to $\tilde{H}_*(|\mathcal{P}|; R)$ and its differential

$$(11) \quad \tilde{H}_n(F_n C_n / F_{n-1} C_n) \rightarrow \tilde{H}_{n-1}(F_{n-1} C_{n-1} / F_{n-2} C_{n-1})$$

is induced by the differential d of $C_*(\Delta(\mathcal{P}); R)$. Hence, the differential of the homology class $[z] \in \tilde{H}_n(F_n C_n / F_{n-1} C_n)$ is $d([z]) = [(-1)^n d_n(z)]$.

3. LOCAL COVERING FAMILIES

Forman's Morse theory [2] for cell complexes runs parallel to classical Morse Theory. For example, the topology of a simplicial complex is determined by the unmatched simplices in an acyclic matching of the Hasse diagram of the complex [1, Proposition 3.3]. A partial converse to this result was proven by Stanley [4, Theorem 1.2]. Here we describe similar notions for a poset that determine the local homotopy type of its nerve.

12. Definition. Let \mathcal{P} be a graded poset. A local covering family \mathcal{K} for \mathcal{P} is a family of subposets $\mathcal{K}_p \subseteq \hat{\mathcal{P}}_{<p}$ containing $\hat{0}$ and maps $\eta_p : \mathcal{K}_p \rightarrow \hat{\mathcal{P}}_{<p} \setminus \mathcal{K}_p$ for all $p \in \mathcal{P}$ such that

- (1) if $q \in \mathcal{K}_p$ then $q < \eta_p(q)$ and $\dim \eta_p(q) = \dim q + 1$,
- (2) if $q \notin \mathcal{K}_p$ and $q \leq p$ then $\eta_p(\hat{0}) = \eta_q(\hat{0})$,
- (3) if $q \in \mathcal{K}_p$, $r \in \mathcal{K}_p$ and $q \leq r$ then $\eta_p(q) \leq \eta_p(r)$, and
- (4) if $q \in \mathcal{K}_p$, $r \notin \mathcal{K}_p$ and $q \leq r \leq p$ then $\eta_p(q) \leq r$.

13. Remark. Note that $(\mathcal{P}_{<p})_{-1} = (\mathcal{K}_p)_{-1} = \{\hat{0}\}$ and by items (1) and (2) it is easy to see that $(\mathcal{P}_{<p})_0 = (\mathcal{K}_p)_0 \cup \{\eta_p(\hat{0})\}$.

14. Example. Let Δ be the inclusion poset of an (abstract) simplicial complex on the set of vertices $V = \{0, \dots, m\}$ and graded by simplicial dimension. We may equip Δ with a local covering family as follows: For the n -simplex $\sigma = \{v_0, \dots, v_n\}$ with $v_0 < \dots < v_n$ define the subposet $\mathcal{K}_\sigma \subseteq \hat{\Delta}_{<\sigma}$ by

$$\mathcal{K}_\sigma = \{\text{simplices } \tau \text{ of } \sigma \text{ such that } v_0 \notin \tau\}.$$

The map η_σ sends $\tau \in \mathcal{K}_\sigma$ to $\eta_\sigma(\tau) = \tau \cup \{v_0\}$.

15. Example. Quillen's complex $\mathcal{A}_\pi(G)$ of a finite group G may be graded by (dimension-1) and can also be equipped with a local covering family: First, give a total order to the order- π subgroups of G , $\mathcal{A}_\pi(G)_0 = \{V_0, \dots, V_m\}$. Then, for $H \in \mathcal{A}_\pi(G)$, set $H^* = \min\{V_i | V_i \leq H\}$ and define

$$\mathcal{K}_H = \{\text{subgroups } I \leq H \text{ such that } H^* \not\leq I\}.$$

The map η_H maps $I \in \mathcal{K}_H$ to the subgroup $\eta_H(I)$ of H generated by I and H^* .

16. Lemma. *Let \mathcal{P} be a graded poset with a local covering family \mathcal{K} . Then \mathcal{P} is locally spherical. For $p \in P$ with $\dim p = n$ the number of $(n-1)$ -spheres in the bouquet $|\mathcal{P}_{<p}|$ is equal to the number of n -simplices $p_0 < p_1 < \dots < p_n = p$ with $\dim p_i = i$ and $p_i \in \mathcal{K}_{p_{i+1}}$ for $i = 0, \dots, n-1$.*

Proof. We prove by induction on $n = \dim p$ that $|\mathcal{P}_{<p}|$ is $(n-1)$ -spherical with the given number of spheres. For $n = 0$ the claim is clear. Let p be of dimension $n \geq 1$ and consider the subposet given by $\mathcal{I} = \mathcal{P}_{<p} \setminus (\mathcal{K}_p)_{n-1}$. We show that \mathcal{I} is conically contractible [3, 1.5] by exhibiting a map of posets $f : \mathcal{I} \rightarrow \mathcal{I}$ such that $x \leq f(x) \leq \eta_p(\hat{0})$ for all x in \mathcal{I} . For this is enough to define $f(x) = x$ if $x < p$, $x \notin \mathcal{K}_p$ and $f(x) = \eta_p(x)$ if $x \in \mathcal{K}_p$. This map f is a map of posets because \mathcal{K}_p is a subposet and because of Definition 12(3) and 12(4). Now by construction the image of f is contained in $\mathcal{P}_{<p} \setminus \mathcal{K}_p$ and hence by Definition 12(2) we have that $f(x) \geq \eta_{f(x)}(\hat{0}) = \eta_p(\hat{0})$ for all x in \mathcal{I} .

As $|\mathcal{I}|$ is contractible, the quotient map $|\mathcal{P}_{<p}| \rightarrow |\mathcal{P}_{<p}|/|\mathcal{I}| \simeq \bigvee_{q \in (\mathcal{K}_p)_{n-1}} \Sigma |\mathcal{P}_{<q}|$ is a homotopy equivalence. By induction, for each $q \in (\mathcal{K}_p)_{n-1}$, the nerve $|\mathcal{P}_{<q}|$ is a bouquet of $(n-2)$ -spheres. Hence, $|\mathcal{P}_{<p}|$ is a bouquet of $(n-1)$ -spheres and the counting formula for the number of $(n-1)$ -spheres in the bouquet holds. \square

17. Corollary. *Let \mathcal{P} be a graded poset with a local covering family \mathcal{K} and let R be a commutative ring with identity. Define inductively the numbers K_n^p for all $p \in \mathcal{P}$ and all $n \in \mathbb{N}$ as follows: $K_0^p = 1$ and $K_{n+1}^p = \sum_{q \in (\mathcal{K}_p)_n} K_n^q$. Then for all p in \mathcal{P} the reduced homology group $\tilde{H}_{\dim p - 1}(|\mathcal{P}_{<p}|; R)$ is free of rank $K_{\dim p}^p$.*

Combining Equation (7) with last corollary we obtain the following result.

18. Theorem. *Let \mathcal{P} be a graded poset which has a local covering family \mathcal{K} and let R be a commutative ring with identity. Then there is a chain complex*

$$\dots \rightarrow \bigoplus_{p \in \mathcal{P}_i} R^{K_i^p} \rightarrow \dots \rightarrow \bigoplus_{p \in \mathcal{P}_1} R^{K_1^p} \rightarrow \bigoplus_{p \in \mathcal{P}_0} R \rightarrow 0$$

whose homology is $H_*(|\mathcal{P}|; R)$.

Note that this chain complex does not depend (up to isomorphism) on the local covering family \mathcal{K} as it is isomorphic to the chain complex (7) coming from the spectral sequence deduced from the grading of \mathcal{P} .

19. Example. Consider a simplicial complex Δ and the local covering family \mathcal{K} of example 14. If σ is of dimension n we have $|(\mathcal{K}_\sigma)_{n-1}| = 1$ and $K_n^\sigma = 1$. We will see later that the associated chain complex of Theorem 18 is the usual simplicial chain complex.

20. Example. For Quillen's complex $\mathcal{A}_\pi(G)$ of a finite group G at a prime p consider the local covering family of example 15. For a subgroup $H \cong C_\pi^{n+1}$ of dimension n we have $|(\mathcal{K}_H)_{n-1}| = \pi^n$ and $K_n^H = \pi^{\frac{n(n+1)}{2}}$.

4. EXPLICIT DIFFERENTIAL

In this section we study in detail the differential of the chain complex of Theorem 18 (Theorem 1 of the Introduction). Fix a graded poset \mathcal{P} with local covering family \mathcal{K} and R a commutative ring with identity. Call d to the differential of the chain

complex from Theorem 18 applied to \mathcal{P} and \mathcal{K} . If $q \in \mathcal{P}$ with $n = \dim q$ we have from Equations (10) and (11) the following commutative diagram:

$$\begin{array}{ccc}
\tilde{H}_n(F_n C_n / F_{n-1} C_n) & \longrightarrow & \tilde{H}_{n-1}(F_{n-1} C_{n-1} / F_{n-2} C_{n-1}) \\
\cong \uparrow & & \cong \uparrow \\
\bigoplus_{\dim p=n} \tilde{H}_{n-1}(|\mathcal{P}_{<p}|; R) & \xrightarrow{d} & \bigoplus_{\dim p=n-1} \tilde{H}_{n-2}(|\mathcal{P}_{<p}|; R) \\
\uparrow & & \uparrow \\
\tilde{H}_{n-1}(|\mathcal{P}_{<q}|; R) & \xrightarrow{d_q^{\mathcal{P}}} & \bigoplus_{p \in (\mathcal{K}_q)_{n-1}} \tilde{H}_{n-2}(|\mathcal{P}_{<p}|; R).
\end{array}$$

The map at the bottom is given by $d_q^{\mathcal{P}} = \bigoplus_{p \in (\mathcal{K}_q)_{n-1}} [(-1)^n t_p]$. If we project from the codomain of $d_q^{\mathcal{P}}$ onto the components with $p \in (\mathcal{K}_q)_{n-1}$ we get another map

$$\tilde{H}_{n-1}(|\mathcal{P}_{<q}|; R) \xrightarrow{d_q^{\mathcal{K}}} \bigoplus_{p \in (\mathcal{K}_q)_{n-1}} \tilde{H}_{n-2}(|\mathcal{P}_{<p}|; R).$$

This map coincides up to sign with the map in homology induced by the homotopy equivalence $|\mathcal{P}_{<q}| \rightarrow |\mathcal{P}_{<q}|/|\mathcal{I}| \simeq \bigvee_{p \in (\mathcal{K}_q)_{n-1}} \Sigma |\mathcal{P}_{<p}|$ described in the proof of Lemma 16 and hence $d_q^{\mathcal{K}}$ is an isomorphism. We record this fact:

21. Lemma. *Let \mathcal{P} be a graded poset with local covering family \mathcal{K} and let R be a commutative ring with identity. For each $q \in \mathcal{P}$, the map $d_q^{\mathcal{K}}$ described above is an isomorphism.*

Now we shall give an explicit description of $d_q^{\mathcal{P}}$. To do so, we first inductively build a basis of $\tilde{H}_{n-1}(|\mathcal{P}_{<q}|; R)$ using the isomorphism $d_q^{\mathcal{K}}$. For $\dim q = 0$ we define \emptyset_q as a generator of $\tilde{H}_{-1}(\emptyset; R) = R = R\emptyset_q$. For $\dim q = 1$ we fix as basis of $\tilde{H}_0(|\mathcal{P}_{<q}|; R) \cong R^{K_q^1}$ the elements $\eta_q(\hat{0}) - p$ for $p \in (\mathcal{K}_q)_0$, see Remark 13. Then $d_q^{\mathcal{K}}(\eta_q(\hat{0}) - p) = \emptyset_p$ for each $p \in (\mathcal{K}_q)_0$ as wished.

Now assume a basis of $\tilde{H}_{n-2}(|\mathcal{P}_{<p}|; R)$ has been constructed for every element p with $\dim p \leq n-1$. We want to construct a basis of $\tilde{H}_{n-1}(|\mathcal{P}_{<q}|; R)$ with $\dim q = n$ and such that $d_q^{\mathcal{K}}$ carries basic elements to basic elements. This is analogue to construct a local homotopy inverse to the homotopy equivalence above, i.e., to suspend the classes from $|\mathcal{P}_{<p}|$.

So fix a basic element $b \in \tilde{H}_{n-2}(|\mathcal{P}_{<p}|; R)$ for some $p \in (\mathcal{K}_q)_{n-1}$. We want $B \in \tilde{H}_{n-1}(|\mathcal{P}_{<q}|; R)$ with $d_q^{\mathcal{K}}(B) = b$. If we write $b = [z]$ and $B = [Z]$ then we are seeking for a class Z such that

$$d(Z) = 0 \text{ and } d_q^{\mathcal{K}}(Z) = z.$$

We will set $Z = Z_1 + Z_2$, where Z_i correspond to a “cone over z ” for $i = 1, 2$, giving a “suspension of z ”.

Set Z_1 as the “suspension at p ”, i.e., $Z_1 = (-1)^n s_p(z)$, where we add the sign for convenience. To define Z_2 note that $z \in C_{n-2}(|\mathcal{I}|)$, where $\mathcal{I} = \mathcal{P}_{<q} \setminus (\mathcal{K}_q)_{n-1}$ and that $|\mathcal{I}|$ is contractible by Lemma 16. Hence, there exists a unique element $Z_2 \in C_{n-1}(|\mathcal{I}|)$ whose differential is z . Now we have

$$d(Z) = d((-1)^n s_p(z)) + d(Z_2) = (-1)^n (s_p(d(z)) + (-1)^{n-1} z) + z = 0$$

and

$$d_q^{\mathcal{K}}(Z) = \bigoplus_{p' \in (\mathcal{K}_q)_{n-1}} (-1)^n ((-1)^n t_{p'}(s_p(z)) + t_{p'}(Z_2)) = z$$

by Equation (9) and because $\mathcal{I} \cap (\mathcal{K}_q)_{n-1} = \emptyset$. We have proven:

22. Lemma. *Let \mathcal{P} be a graded poset with local covering family \mathcal{K} and let R be a commutative ring with identity. There are basis of $\tilde{H}_{\dim q-1}(|\mathcal{P}_{<q}|; R)$ for all $q \in \mathcal{P}$ such that all isomorphisms $d_q^\mathcal{K}$ carry basic elements to basic elements.*

23. Example. We compute the differential of Example 5. First, the numbers $K_{\dim p}^p$ are easily computed to be

$$K_0^f = K_0^g = K_0^h = 1, K_1^c = K_1^d = K_1^e = 2 \text{ and } K_2^a = K_2^b = 4.$$

In dimension 0 we have the generators \emptyset_f, \emptyset_g and \emptyset_h . For dimension 1, we have, for instance, the chain $g < c$ with $g \in \mathcal{K}_c$. Its associated generator is $f - g$. The signed truncations of $f - g$ are $-t_f(f - g) = -\emptyset_f$, $-t_g(f - g) = \emptyset_g$ and $-t_h(f - g) = 0$ and hence

$$d_c^{\mathcal{P}}(g - f) = -\emptyset_f \oplus \emptyset_g \oplus 0.$$

Analogously, we must consider the chain $h < c$ with generator $f - h$ and, for objects d and e , the generators $g - f$, $g - h$ and $h - f$, $h - g$ respectively.

Now, consider a with $\dim a = 2$ and the chain $f < d < a$ with $f \in \mathcal{K}_d$ and $d \in \mathcal{K}_a$. We lift the basic element $b = [z] = z = g - f$. On the one hand, we have

$$Z_1 = (-1)^2 s_d(g - f) = g < d - f < d.$$

To construct Z_2 we use that the subposet $\{c, f, g, h\}$ is contractible. By inspection we have $Z_2 = f < c - g < c$ as

$$d(Z_2) = c - f - c + g = g - f = z.$$

So we have $Z_1 + Z_2 = g < d - f < d + f < c - g < c$ and the truncations $t_c(Z_1 + Z_2) = f - g$, $t_d(Z_1 + Z_2) = g - f$ and $t_e(Z_1 + Z_2) = 0$. This gives for $B = [Z_1 + Z_2]$ the formula

$$d_a^{\mathcal{P}}(B) = (f - g) \oplus (g - f) \oplus 0.$$

An analogous computations must be made for the chains $h < d < a$, $f < e < a$, $g < e < a$, $g < c < b$, $h < c < b$, $f < d < b$ and $h < d < b$. Summing up, the differentials $\mathbb{Z}^8 \rightarrow \mathbb{Z}^6$ and $\mathbb{Z}^6 \rightarrow \mathbb{Z}^3$ are given by the matrices

$$\begin{pmatrix} 1 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix}.$$

From here we deduce $H_2(|\mathcal{P}|; \mathbb{Z}) = \mathbb{Z}^4$, $H_1(|\mathcal{P}|; \mathbb{Z}) = 0$ and $H_0(|\mathcal{P}|; \mathbb{Z}) = \mathbb{Z}$.

24. Example. Consider a simplicial complex Δ and the local covering family \mathcal{K} of Example 14. For the n -simplex $\sigma = \{v_0, \dots, v_n\}$ with $v_0 < \dots < v_n$ there is just one sphere in the bouquet $|\Delta_{<\sigma}|$ as $K_n^\sigma = 1$ (Example 19). It corresponds to the inclusions of simplices $v_0 < \{v_0, v_1\} < \dots < \{v_0, v_1, \dots, v_n\}$ (cf. Lemma 16). Denote by $b_\sigma = [z_\sigma]$ the basic element constructed by the procedure described above and starting with $z_{v_0} = -\emptyset_{v_0}$ (opposite sign to the construction above just for convenience). We claim that

$$z_\sigma = \sum_{i=0}^n (-1)^i s_{d_i(\sigma)}(z_{d_i(\sigma)})$$

for $n \geq 1$ and we prove it by induction. For $n = 1$ and $\sigma = \{v_0, v_1\}$ we have $z_\sigma = v_0 - v_1 = s_{v_0}(\emptyset_{v_0}) - s_{v_1}(\emptyset_{v_1}) = s_{v_1}(z_{v_1}) - s_{v_0}(z_{v_0})$. For the inductive step, consider

$\sigma = \{v_0, \dots, v_n\}$. To construct $Z = z_\sigma$ we must suspend $z = z_{d_0(\sigma)} = z_{\{v_1, \dots, v_n\}}$ by means of two cones $Z = Z_1 + Z_2$, where $Z_1 = (-1)^n s_{d_0(\sigma)}(z)$ and Z_2 is the unique element with $d(Z_2) = z$. So it is enough to see that $Z_2 = \sum_{i=1}^n (-1)^i s_{d_i(\sigma)}(z_{d_i(\sigma)})$:

$$d(Z_2) = \sum_{i=1}^n (-1)^i d(s_{d_i(\sigma)}(z_{d_i(\sigma)})) = \sum_{i=1}^n (-1)^i (s_{d_i(\sigma)}(d(z_{d_i(\sigma)})) + (-1)^{n-1} z_{d_i(\sigma)}),$$

where in the last equality we have used Equation (9). Now, the element $d(z_{d_i(\sigma)})$ is zero as $z_{d_i(\sigma)}$ is a cycle. Using the induction hypothesis we get

$$d(Z_2) = \sum_{i=1}^n \sum_{j=0}^{n-1} (-1)^{i+j} s_{d_j(d_i(\sigma))}(z_{d_j(d_i(\sigma))}),$$

which, by the simplicial identities, equals

$$\sum_{k=0}^{n-1} (-1)^k s_{d_{k-1}(d_0(\sigma))}(z_{d_{k-1}(d_0(\sigma))}).$$

By the induction hypothesis this expression is exactly $z_{d_0(\sigma)} = z$. From the expression for z_σ and (9) it is clear that $d_\sigma^\Delta(z_\sigma) = \oplus_{i=0}^n (-1)^i z_{d_i(\sigma)}$ and hence

$$(25) \quad d_\sigma^\Delta(b_\sigma) = \oplus_{i=0}^n (-1)^i b_{d_i(\sigma)},$$

i.e., we recover the simplicial differential. This shows that Theorem 2 of the introduction holds. To illustrate the construction we reproduce it for $\sigma = 0 < 1 < 2$. Recall that we want to suspend the basic element $1 - 2$ and in this case we have:

$$Z_1 = 1 < 12 - 2 < 12 \text{ and } Z_2 = 0 < 01 - 1 < 01 - 0 < 02 + 2 < 02.$$

Thus $Z = Z_1 + Z_2 = s_{12}(0 - 2) - s_{02}(0 - 2) + s_{01}(0 - 1)$, $d(Z) = 0$ and we have the truncations

$$t_{12}(Z) = 1 - 2, t_{02}(Z) = 2 - 0 \text{ and } t_{01}(Z) = 0 - 1.$$

26. Example. For Quillen's complex $\mathcal{A}_\pi(G)$ of a finite group G at a prime π consider the local covering family of Example 15. For a subgroup $H \cong C_\pi^{n+1}$ of dimension $(n+1)$ we have $K_n^H = \pi^{\frac{n(n+1)}{2}}$ $(n-1)$ -spheres in $|\mathcal{A}_\pi(G)_{<H}|$ (Example 20). Denote by $V_0 < \dots < V_m$ the ordered rank 1 subgroups of H (restriction of the order in $\mathcal{A}_\pi(G)_0$) and for any subgroup $W \leq H$ set $W^* = \min\{V_i | V_i \leq W\}$. Then these spheres are indexed by chain of subgroups $W_0 < W_1 < \dots < W_{n-1} < W_n = H$ where W_i does not contain W_{i+1}^* for $i = 0, \dots, n-1$ (cf. Lemma 16). It is clear that any such chain can be written as

$$(27) \quad W_0 = W_0^* < W_0^* \times W_1^* < \dots < W_0^* \times W_1^* \times \dots \times W_{n-1}^* \times W_n^* = H.$$

Moreover, this chain is determined by the ordered cartesian product

$$(28) \quad W_0^* \times W_1^* \times \dots \times W_{n-1}^* \times W_n^*,$$

where we assume that the factors are arranged in decreasing order as rank-1 subgroups of H , i.e., $W_n^* < W_{n-1}^* < \dots < W_2^* < W_1^* < W_0^*$. Denote by $b_{W_0^* \times \dots \times W_n^*} = [z_{W_0^* \times \dots \times W_n^*}]$ the basic element associated to the chain (27), and which is unambiguously determined by the ordered cartesian product (28).

We start the induction with $z_{V_i} = -\emptyset_{V_i}$. Notice that the subposet of $\mathcal{A}_\pi(G)$ with objects the subgroups $W_{i_0}^* \times W_{i_1}^* \times \dots \times W_{i_l}^*$ with $i_0 < \dots < i_l$ and $0 \leq l \leq n$ is isomorphic to the n -simplex, i.e., to the full poset of subgroups of the set

$\{0, 1, \dots, n\}$. Using this fact and the same argument as in Example 24 one can prove that:

$$z_{W_0^* \times \dots \times W_n^*} = \sum_{i=0}^n (-1)^i s_{W_0^* \times W_{i-1}^* \times W_{i+1}^* \dots \times W_n^*} (z_{W_0^* \times W_{i-1}^* \times W_{i+1}^* \dots \times W_n^*})$$

and

$$d_H^{\mathcal{A}^\pi(G)}(b_{W_0^* \times \dots \times W_n^*}) = \oplus_{i=0}^n (-1)^i b_{W_0^* \times W_{i-1}^* \times W_{i+1}^* \dots \times W_n^*}.$$

So the differential $d^{\mathcal{A}^\pi(G)}$ behaves locally like the simplicial differential d^Δ . For instance, for $H \cong C_\pi^2$ with rank 1 subgroups $V_0 < V_1 < V_2 < \dots < V_\pi$, we have

$$d_H^{\mathcal{A}^\pi(G)}(b_{V_i \times V_0}) = b_{V_0} \oplus (-b_{V_i})$$

for the π 1-spheres associated to the chains $V_i < V_i \times V_0$, $i = 1, \dots, \pi$.

5. FREE OBJECTS

The next result proves Theorem 6 of the introduction:

29. Theorem. *Let \mathcal{P} be a locally π -Quillen poset at the prime π with $n = \max \dim \mathcal{P}$. If the homology group $H_n(|\mathcal{P}|; R)$ is zero then proportion r of free objects among the non-maximal objects of \mathcal{P}_{n-1} satisfies:*

$$r \geq \frac{\pi^{n+1} - 2\pi^n + 1}{\pi^{n+1} - \pi^n}.$$

Proof. By Theorem 18 there is a chain complex

$$0 \rightarrow R^{|\mathcal{P}_n| \pi^{\frac{n(n+1)}{2}}} \xrightarrow{d_n} R^{|\mathcal{P}_{n-1}| \pi^{\frac{n(n-1)}{2}}} \rightarrow \dots \rightarrow R^{|\mathcal{P}_2| \pi^3} \rightarrow R^{|\mathcal{P}_1| \pi} \rightarrow R^{|\mathcal{P}_0|} \rightarrow 0$$

whose homology is $H_*(|\mathcal{P}|; R)$. Write $\mathcal{P}_{n-1} = \mathcal{P}' \cup \mathcal{P}''$ where the objects of \mathcal{P}' are maximal in \mathcal{P} and the objects in \mathcal{P}'' are not. So, if for $p \in \mathcal{P}_{n-1}$ we set n_p to be the number of maximal subgroups q of \mathcal{P}_n with $q > p$, we have $n_p = 0$ for $p \in \mathcal{P}'$ and $n_p \geq 1$ for $p \in \mathcal{P}''$. As $H_n(|\mathcal{P}|; R) = 0$ the differential d_n must be injective. In particular, we must have the inequality

$$(30) \quad |\mathcal{P}_n| \pi^{\frac{n(n+1)}{2}} \leq |\mathcal{P}''| \pi^{\frac{(n-1)n}{2}}.$$

The number of edges in \mathcal{P} among objects of \mathcal{P}_n and objects of \mathcal{P}_{n-1} is

$$(31) \quad |\mathcal{P}_n| \frac{\pi^{n+1} - 1}{\pi - 1} = \sum_{p \in \mathcal{P}_{n-1}} n_p = \sum_{k=1}^K k |N_k|,$$

where $N_k = \{p \in \mathcal{P}'' | n_p = k\}$ and $K = \max\{n_p | p \in \mathcal{P}''\}$. As $|\mathcal{P}''| = \sum_{k=1}^K |N_k|$ we get from Equations (30) and (31) that

$$(32) \quad \sum_{k=1}^K k |N_k| \leq f(\pi, n) \left(\sum_{k=1}^K |N_k| \right)$$

where $f(\pi, n) = \frac{\pi^{n+1} - 1}{\pi^n(\pi - 1)}$. Note that $\lim_{n \rightarrow \infty} f(\pi, n) = \frac{\pi}{\pi - 1}$ and $\lim_{\pi \rightarrow \infty} f(\pi, n) = 1^+$. Now using the inequalities $k |N_k| > 2 |N_k|$ for $k \geq 3$ we obtain from Equation 32 that

$$(33) \quad \frac{|N_1|}{|\mathcal{P}''|} \geq g(\pi, n)$$

with $g(\pi, n) = 2 - f(\pi, n) = \frac{\pi^{n+1} - 2\pi^n + 1}{\pi^n(\pi - 1)}$. Note that $\lim_{n \rightarrow \infty} g(\pi, n) = \frac{\pi - 2}{\pi - 1}$ and $\lim_{\pi \rightarrow \infty} g(\pi, n) = 1^-$. The number $\frac{|N_1|}{|P^n|}$ is the ratio r in the statement. \square

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